Fuzzy Conditional Inference under Max-Composition

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ABSTRACT

This paper shows that the majority of fuzzy inference methods for a fuzzy conditional proposition "If \( x \) is \( A \) then \( y \) is \( B \)," with \( A \) and \( B \) fuzzy concepts, can infer very reasonable consequences which fit our intuition with respect to several criteria such as *modus ponens* and *modus tollens*, if a new composition called "max-\( \circ \) composition" is used in the compositional rule of inference, though reasonable consequences cannot always be obtained when using the max-min composition, which is used usually in the compositional rule of inference. Furthermore, it is shown that a syllogism holds for the majority of the methods under the max-\( \circ \) composition, though they do not always satisfy the syllogism under the max-min composition.

1. INTRODUCTION

In our daily life we often make inferences of the form

\[
\begin{align*}
\text{Ant 1:} & \quad \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\
\text{Ant 2:} & \quad x \text{ is } A'. \\
\text{Cons:} & \quad y \text{ is } B'.
\end{align*}
\]

where \( A, A', B, \) and \( B' \) are fuzzy concepts. In order to make such an inference with fuzzy concepts, Zadeh [1] suggested an inference rule called the "compositional rule of inference," which infers \( B' \) of Cons from Ant 1 and Ant 2 by taking the max-min composition of \( A' \) and the fuzzy relation which is translated from the fuzzy conditional proposition "If \( x \) is \( A \) then \( y \) is \( B \)." In this connection, he [1], Mamdani [2], and Mizumoto et al. [3–7] suggested several translating rules for translating the fuzzy proposition "If \( x \) is \( A \) then \( y \) is \( B \)" into a fuzzy relation.

In [4–6] we pointed out that the consequences inferred by Zadeh's and Mamdani's methods do not always fit our intuition, and proposed some new methods which can lead to consequences coinciding with our intuition with respect to several criteria, such as *modus ponens* and *modus tollens*. Moreover, we suggested in [7] new translating rules which are obtained by introducing...
implication rules of many valued logic systems, but these methods were found not to infer reasonable consequences.

In [8], however, we have shown that, although the translating rule called by Zadeh the "arithmetic rule" does not infer reasonable consequences in the compositional rule of inference which uses the max-min composition, the arithmetic rule can infer very reasonable consequences when a new composition named "max-\(\ominus\) composition" is used in the compositional rule of inference, where \(\ominus\) is the operation of "bounded product," which is dual to the "bounded sum" introduced by Zadeh [1].

As continuation of our study [8], this paper investigates the inference results of all the translating rules proposed until now under the max-\(\ominus\) composition, and shows that the majority of the translating rules can infer very reasonable consequences which fit our intuition. Moreover, it is shown that the majority of the translating rules satisfy a syllogism under the max-\(\ominus\) composition.

2. TRANSLATING RULES

We shall first consider the following form of inference in which a fuzzy conditional proposition is contained:

\[
\begin{align*}
\text{Ant 1: } & \text{ If } x \text{ is } A \text{ then } y \text{ is } B. \\
\text{Ant 2: } & x \text{ is } A'. \\
\text{Cons: } & y \text{ is } B'.
\end{align*}
\]  \(1\)

where \(x\) and \(y\) are the names of objects, and \(A, A', B,\) and \(B'\) are fuzzy concepts represented by fuzzy sets in universes of discourse \(U, U, V,\) and \(V,\) respectively. This form of inference may be viewed as fuzzy modus ponens, which reduces to the classical modus ponens when \(A' = A\) and \(B' = B.\)

Moreover, the following form of inference is possible, which also contains a fuzzy conditional proposition:

\[
\begin{align*}
\text{Ant 1: } & \text{ If } x \text{ is } A \text{ then } y \text{ is } B. \\
\text{Ant 2: } & y \text{ is } B'. \\
\text{Cons: } & x \text{ is } A'.
\end{align*}
\]  \(2\)

This inference can be considered as fuzzy modus tollens, which reduces to the classical modus tollens when \(B' = \text{not } B\) and \(A' = \text{not } A.\)

The fuzzy proposition "If \(x\) is \(A\) then \(y\) is \(B\)" in (1) and (2) may represent a certain relationship between \(A\) and \(B.\) From this point of view, a number of translating rules have been proposed for translating the fuzzy conditional proposition "If \(x\) is \(A\) then \(y\) is \(B\)" into a fuzzy relation in \(U \times V.\)
FUZZY CONDITIONAL INFERENCE

Let \( A \) and \( B \) be fuzzy sets in \( U \) and \( V \), respectively, which are represented as

\[
A = \int_U \mu_A(u) / u, \quad B = \int_V \mu_B(v) / v,
\]

and let \( \land, \lor, \neg \), and \( \oplus \) be the cartesian product, union, intersection, complement, and bounded sum for fuzzy sets, respectively. Then the following fuzzy relations in \( U \times V \) are translations of the fuzzy conditional proposition "If \( x \) is \( A \) then \( y \) is \( B \)." \( \text{Rm} \) (maximin rule) and \( \text{Ra} \) (arithmetic rule) were proposed by Zadeh [1], \( \text{Rc} \) (min rule) by Mamdani [2], and the others were created by Mizumoto et al. [3–7] by introducing the implications of many valued logic systems [9–11].

\[
\text{Rm} = (A \times B) \cup (-A \times V)
\]
\[
= \int_{U \times V} \left[ \mu_A(u) \land \mu_B(v) \right] \lor \left[ 1 - \mu_A(u) \right] / (u, v); \tag{3}
\]

\[
\text{Ra} = (-A \times V) \oplus (U \times B)
\]
\[
= \int_{U \times V} \land \left( 1 - \mu_A(u) + \mu_B(v) \right) / (u, v); \tag{4}
\]

\[
\text{Rc} = A \times B
\]
\[
= \int_{U \times V} \mu_A(u) \land \mu_B(v) / (u, v); \tag{5}
\]

\[
\text{Rg} = A \times V \Rightarrow U \times B
\]
\[
= \int_{U \times V} \left[ \mu_A(u) \rightarrow g \mu_B(v) \right] / (u, v), \tag{6}
\]

where

\[
\mu_A(u) \rightarrow g \mu_B(v) = \begin{cases} 
1, & \mu_A(u) \leq \mu_B(v), \\
0, & \mu_A(u) > \mu_B(v); 
\end{cases}
\]

\[
\text{Rg} = A \times V \Rightarrow U \times B
\]
\[
= \int_{U \times V} \left[ \mu_A(u) \rightarrow g \mu_B(v) \right] / (u, v), \tag{7}
\]
where

\[
\mu_A(u) \to \mu_B(v) = \begin{cases} 
1, & \mu_A(u) \leq \mu_B(v), \\
\mu_B(v), & \mu_A(u) > \mu_B(v);
\end{cases}
\]

\[
R_{sg} = (A \times V \Rightarrow U \times B) \cap (\neg A \times V \Rightarrow U \times \neg B)
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \to \mu_B(v) \right] \land \left[ 1 - \mu_A(u) \to 1 - \mu_B(v) \right] / (u, v); \quad (8)
\]

\[
R_{gg} = (A \times V \Rightarrow U \times B) \cap (\neg A \times V \Rightarrow U \times \neg B)
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \to \mu_B(v) \right] \land \left[ 1 - \mu_A(u) \to 1 - \mu_B(v) \right] / (u, v); \quad (9)
\]

\[
R_{gs} = (A \times V \Rightarrow U \times B) \cap (\neg A \times V \Rightarrow U \times \neg B)
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \to \mu_B(v) \right] \land \left[ 1 - \mu_A(u) \to 1 - \mu_B(v) \right] / (u, v); \quad (10)
\]

\[
R_{ss} = (A \times V \Rightarrow U \times B) \cap (\neg A \times V \Rightarrow U \times \neg B)
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \to \mu_B(v) \right] \land \left[ 1 - \mu_A(u) \to 1 - \mu_B(v) \right] / (u, v); \quad (11)
\]

\[
R_b = (\neg A \times V) \cup (U \times B)
\]

\[
= \int_{U \times V} [1 - \mu_A(u)] \lor \mu_B(v) / (u, v). \quad (12)
\]

\[
R_\Delta = A \times V \Rightarrow U \times B
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \to \mu_B(v) \right] / (u, v), \quad (13)
\]

where

\[
\mu_A(u) \to \mu_B(v) = \begin{cases} 
1, & \mu_A(u) \leq \mu_B(v), \\
\mu_B(v) / \mu_A(u), & \mu_A(u) > \mu_B(v);
\end{cases}
\]

\[
R_\Delta = A \times V \Rightarrow U \times B
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \to \mu_B(v) \right] / (u, v), \quad (14)
\]
where

\[
\mu_A(u) \rightarrow_D \mu_B(v) = \left[ \mu_A(u) \rightarrow \mu_B(v) \right] \land \left[ 1 - \mu_B(v) \rightarrow \nabla \mu_A(u) \right]
\]

\[
= \begin{cases} 
1 \land \frac{\mu_B(v)}{\mu_A(u)} \land \frac{1 - \mu_A(u)}{1 - \mu_B(v)}, & \mu_A(u) > 0, \ 1 - \mu_B(v) > 0, \\
1, & \mu_A(u) = 0 \text{ or } 1 - \mu_B(v) = 0;
\end{cases}
\]

\[
R_\ast = A \times V \Rightarrow U \times B
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \rightarrow_D \mu_B(v) \right] / (u, v),
\]

where

\[
\mu_A(u) \rightarrow_D \mu_B(v) = 1 - \mu_A(u) + \mu_A(u) \mu_B(v);
\]

\[
R_\# = A \times V \Rightarrow U \times B
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \rightarrow_\# \mu_B(v) \right] / (u, v),
\]

where

\[
\mu_A(u) \rightarrow_\# \mu_B(v) = \left[ \mu_A(u) \land \mu_B(v) \right] \lor \left[ 1 - \mu_A(u) \land 1 - \mu_B(v) \right]
\]

\[
\lor \left[ \mu_B(v) \land 1 - \mu_A(u) \right]
\]

\[
= \left[ 1 - \mu_A(u) \lor \mu_B(v) \right] \land \left[ \mu_A(u) \lor 1 - \mu_A(u) \right]
\]

\[
\land \left[ \mu_B(v) \lor 1 - \mu_B(v) \right];
\]

\[
R_\Box = A \times V \Rightarrow U \times B
\]

\[
= \int_{U \times V} \left[ \mu_A(u) \rightarrow_\Box \mu_B(v) \right] / (u, v),
\]

where

\[
\mu_A(u) \rightarrow_\Box \mu_B(v) = \begin{cases} 
1, & \mu_A(u) < 1 \text{ or } \mu_B(v) = 1, \\
0, & \mu_A(u) = 1, \ 1 - \mu_B(v) < 1.
\end{cases}
\]
We shall next review the properties of the "bounded product" \( \odot \) in order to define a composition, called "max-\( \odot \) composition," which is used in the compositional rule of inference.

The operation of bounded product \( \odot \) is defined as follows: For any \( x, y \in [0, 1] \),

\[
x \odot y = 0 \lor (x + y - 1).
\]  

(18)

This is a dual operation of the "bounded sum" \( \oplus \) introduced by Zadeh [1]:

\[
x \oplus y = 1 \land (x + y).
\]  

(19)

For the bounded product \( \odot \), the following properties are obtained. The properties of the bounded sum \( \oplus \) are omitted, since it is dual to \( \odot \). More detailed properties of \( \odot \) and \( \oplus \) are found in [12–14].

\[
x \leq y, \quad z \leq w \quad \Rightarrow \quad x \odot z \leq y \odot w,
\]

\[
x \odot x \leq x,
\]

\[
x \odot y = y \odot x,
\]

\[
x \odot (y \odot z) = (x \odot y) \odot z,
\]

\[
x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z),
\]

\[
x \odot (1 - x) = (1 - x) \oplus (1 - y),
\]

\[
x \odot 1 = x, \quad x \odot 0 = 0,
\]

\[
x \odot (1 - x) = 0.
\]

Moreover, the following properties are also obtained by combining \( \odot \) with \( \lor \) and \( \land \):

\[
x \odot (y \lor z) = (x \odot y) \lor (x \odot z),
\]

\[
x \odot (y \land z) = (x \odot y) \land (x \odot z),
\]

\[
x \lor (y \odot z) \geq (x \lor y) \odot (x \lor z),
\]

\[
x \land (y \odot z) \geq (x \land y) \odot (x \land z).
\]
FUZZY CONDITIONAL INFECTION

Using the bounded product $\circ$, we can easily define the \textit{max-$\circ$ composition} of a fuzzy set $A$ in $U$ and a fuzzy relation $R$ in $U \times V$:

$$A \circ R \Leftrightarrow \mu_{A \circ R}(v) = \bigvee_u \{ \mu_A(u) \circ \mu_R(u, v) \}. \quad (20)$$

From the definition of max-$\circ$ composition $\circ$, we have the following properties, which may be useful in discussing the fuzzy conditional inference.

Let $A$, $A_1$, and $A_2$ be fuzzy sets in $U$, and $R$, $R_1$, and $R_2$ be fuzzy relations in $U \times V$. Then

$$A \circ (R_1 \cup R_2) = (A \circ R_1) \cup (A \circ R_2),$$

$$(A_1 \cup A_2) \circ R = (A_1 \circ R) \cup (A_2 \circ R),$$

$$A \circ (R_1 \cap R_2) \subseteq (A \circ R_1) \cap (A \circ R_2),$$

$$(A_1 \cap A_2) \circ R \subseteq (A_1 \circ R) \cap (A_2 \circ R).$$

Now we shall begin with the fuzzy \textit{modus ponens} of (1). Using the max-$\circ$ composition (20), we can obtain the consequence $B'$ of Cons in (1) from Ant 1 and Ant 2 by taking the max-$\circ$ composition $\circ$ of the fuzzy set $A'$ and the fuzzy relation given in (3)–(17). For example, we can have

$$Bm' = A' \circ Rm$$

$$= A' \circ [(A \times B) \cup (\neg A \times V)]. \quad (21)$$

The membership function of the fuzzy set $Bm'$ in $V$ is given as

$$\mu_{Bm'}(v) = \bigvee_u \{ \mu_{A'}(u) \circ \mu_{Rm}(u, v) \}$$

$$= \bigvee_u \{ \mu_{A'}(u) \circ (\mu_A(u) \land \mu_B(v)) \lor [1 - \mu_A(u)] \}. \quad (22)$$

In the same way, we have

$$Ba' = A' \circ Ra = A' \circ [(-A \times V) \oplus (U \times B)], \quad (23)$$

$$Bc' = A' \circ Rc = A' \circ (A \times B), \quad (24)$$

$$Bs' = A' \circ Rs = A' \circ [A \times V = U \times B], \quad (25)$$

$$\vdots$$
Similarly, in the fuzzy *modus tollens* of (2), the consequence \( A' \) in Cons can be deduced using the composition \( \square \) of the fuzzy relation and the fuzzy set \( B' \). Namely,

\[
A' = R m \square B' \\
= [(A \times B) \cup (\neg A \times V)] \square B' \\
= \int_U \bigvee_v \left( (\mu_A(u) \land \mu_{B'}(v)) \lor \left[ 1 - \mu_A(u) \right] \right) \land \mu_{B'}(v) / u,
\]

\[A' = R a \square B' = [(-A \times V) \oplus (U \times B)] \square B', \]  \( (27) \)

\[A' = R c \square B' = (A \times B) \square B', \]  \( (28) \)

\[A' = R s \square B' = \left[ A \times V \Rightarrow U \times B \right] \square B', \]  \( (29) \)

\[\vdots\]

3. COMPARISON OF FUZZY INference METHODS UNDER MAX-\( \odot \) COMPOSITION

In this section we shall make comparisons of the fuzzy inference methods obtained above by applying 15 fuzzy relations of (3)–(17) to the fuzzy *modus ponens* (1) and the fuzzy *modus tollens* (2).

In the fuzzy *modus ponens*, we shall show what the consequences \( Bm' \), \( Ba' \), \( Bc' \),...will be when using the max-\( \odot \) composition [as in (21)–(25)] of the fuzzy set \( A' \) and the fuzzy relation, where the fuzzy set \( A' \) is

\[A' = A = \int_U \mu_A(u) / u,\]

\[A' = \text{very} A = A^2 = \int_U \mu_A(u)^2 / u,\]

\[A' = \text{more or less} \ A = \sqrt[A]{A} = \int_U \sqrt[\mu_A(u)] / u,\]

\[A' = \text{not} \ A = \neg A = \int_U 1 - \mu_A(u) / u,\]

which are typical examples of \( A' \).

Similarly, in the fuzzy *modus tollens* we shall show what the consequences \( Am' \), \( Aa' \), \( Ac' \),...will be when using the max-\( \odot \) composition [as in (26)–(29)] of
the fuzzy relation and the fuzzy set \( B' \), where \( B' \) is

\[
B' = \text{not } B = \neg B = \int_V 1 - \mu_B(v) / v,
\]

\[
B' = \text{not very } B = \neg B^2 = \int_V 1 - \mu_B(v)^2 / v,
\]

\[
B' = \text{not more or less } B = -\sqrt{B} = \int_V 1 - \sqrt{\mu_B(v)} / v,
\]

\[
B' = B = \int_V \mu_B(v) / v.
\]

We shall begin with the fuzzy modus ponens in (1). We shall assume in the discussion of the fuzzy modus ponens that \( \mu_A(u) \) takes all values in \([0, 1]\) as \( u \) varies over all of \( U \), that is, \( \mu_A \) is a function onto \([0, 1]\). Clearly, from the assumption, the fuzzy set \( A \) is a normal fuzzy set.

We shall first discuss Rm and obtain Bm' of (21). From the above assumption, the expression (22) can be rewritten as

\[
b_m' = \bigvee_x \{ x' \odot [(x \land b) \lor (1 - x)] \}, \tag{30}
\]

and

\[
f(x) = x' \odot [(x \land b) \lor (1 - x)] \tag{31}
\]

by letting

\[
\mu_A(u) = x, \quad \mu_A'(u) = x', \quad \mu_B(v) = b, \quad \mu_{Bm'}(v) = b_m'. \tag{32}
\]

(i) For \( A' = A \): When \( A' \) is equal to \( A \) (i.e., \( \mu_{A'} = \mu_A \)), \( x' \) becomes \( x \) from (32). Thus, we have \( f(x) \) of (31) as

\[
f(x) = x \odot [(x \land b) \lor (1 - x)]
\]

\[
= 0 \lor (x + [(x \land b) \lor (1 - x)] - 1)
\]

\[
= 0 \lor ((x - 1 + (x \land b)) \lor [x - 1 + 1 - x])
\]

\[
= 0 \lor (((x - 1 + x) \land (x - 1 + b)) \lor 0)
\]

\[
= 0 \lor [(2x - 1) \land (x - 1 + b)]
\]

\[
= [0 \lor (2x - 1)] \land [0 \lor (x - 1 + b)]. \tag{33}
\]

\[\text{For any real numbers } x, y, \text{ and } z, \text{ we have in general } x + (y \land z) = (x + y) \land (x + z), \]

\[x + (y \lor z) = (x + y) \lor (x + z), \quad (x \land y) \land z = (x - z) \land (y - z), \quad (x \lor y) - z = (x - z) \lor (y - z), \quad x - (y \land z) = (x - y) \lor (x - z), \quad x - (y \lor z) = (x - y) \land (x - z),\]
Figure 1(a) shows partial plots of the expressions $0 \lor (2x - 1)$ and $0 \lor (x - 1 + b)$ of (33) with $b$ as parameter. When $b$ is equal to, say, 0.2, $f(x)$ is indicated by the broken line, and thus $b'_m = \lor_x f(x)$ of (30) at $b = 0.2$ is seen to be 0.2 by observing the maximum of this line. In the same way, at $b = 0.6$, $f(x)$ is shown by the dot-dash line, whose maximum value is 0.6. Thus we have $b'_m = 0.6$ at $b = 0.6$. In general, we can have $b'_m = b$ for any $b$, that is, $b'_m = b$ at $x' = x$, which leads to $\mu_{Bm'} = \mu_B$ at $\mu_{A'} = \mu_A$ from (32). Thus, $Bm' = B$ at $A' = A$. Therefore, from (21),

$$A \square Rm = B,$$

(34)

\[\text{(a) } f(x) \text{ of (33)} \quad \text{(b) } f(x) \text{ of (35)}\]

\[\text{(c) } f(x) \text{ of (37)}\]

Fig. 1. $f(x) = x' \odot [(x \land b) \lor (1 - x)]$ at $x' = x, x^2, \text{ and } \sqrt{x}$. 

which indicates that the *modus ponens* is satisfied by the method Rm under the max-$\odot$ composition $\Box$. It is noted that Rm does not satisfy the *modus ponens* under the max-min composition $[4]$.

(ii) For $A' = \text{very} A$: When $A' = \text{very} A$ ($= A^2$), $x'$ becomes $x^2$. Thus, (31) will be

\[
f(x) = x^2 \odot [(x \land b) \lor (1 - x)] \\
= 0 \lor (x^2 + [(x \land b) \lor (1 - x)] - 1) \\
= 0 \lor [(x^2 - 1 + (x \land b)) \lor [(x^2 - 1 + 1 - x)]] \\
= 0 \lor [(x^2 - 1 + x) \land (x^2 - 1 + b)] \lor (x^2 - x) \\
= 0 \lor [(x^2 + x - 1) \land (x^2 - 1 + b)] \quad \text{since} \quad x^2 - x \leq 0 \\
= [0 \lor (x^2 + x - 1)] \land [0 \lor (x^2 - 1 + b)]. \quad (35)
\]

In Figure 1(b), the expressions $0 \lor (x^2 + x - 1)$ and $0 \lor (x^2 - 1 + b)$ are plotted with $b$ as parameter. For example, at $b = 0.2$, $f(x)$ is shown by the broken line, and its maximum value is 0.2. Thus, $b_m' = \land_x f(x) = 0.2$. When $b = 0.7$, we have $b_m' = 0.7$. Thus, in general, we can obtain $b_m' = b$ for any $b$. Therefore, $Bm' = B$ at $A' = \text{very} A$. Thus,

\[
\text{very} A \Box \text{Rm} = B. \quad (36)
\]

(iii) For $A' = \text{more or less} A$: Since $x' = \sqrt{x}$, $f(x)$ is given by

\[
f(x) = \sqrt{x} \odot [(x \land b) \lor (1 - x)] \\
= (0 \lor (\sqrt{x} + x - 1)) \land (0 \lor (\sqrt{x} - 1 + b)) \lor (\sqrt{x} - x). \quad (37)
\]

In Figure 1(c), $f(x)$ at $b = 0.2$ ($\leq 0.25$) is shown by the broken line, whose maximum value is equal to the maximum value of $\sqrt{x} - x$. The expression $\sqrt{x} - x$ in fact takes its maximum value 0.25 at $x = 0.25$. Thus, we have $b_m' = \land_x f(x) = 0.25$ at $b = 0.2$. It is found from this figure that $b_m' = 0.25$ so long as $b \leq 0.25$. On the other hand, when $b = 0.6$ ($\geq 0.25$), $f(x)$ is indicated by the dot-dash line. Its maximum value is equal to 0.6. In general, we can obtain
\[ b'_m = b \text{ as long as } b \geq 0.25. \text{ Thus we conclude that} \]

\[
b'_m = \begin{cases} 
\frac{1}{4}, & b \leq \frac{1}{4}, \\
q, & b \geq \frac{1}{4}, 
\end{cases}
\]

\[= \frac{1}{4} \lor b.\]

Therefore,

\[\text{more or less } A \Box R_m = B',\]

where

\[\mu_{B'} = \frac{1}{4} \lor \mu_B.\] (38)

(iv) For \( A' = \text{not } A \): Since \( x' = 1 - x \), \( f(x) \) will be

\[f(x) = (1 - x) \odot [(x \land b) \lor (1 - x)] \]

\[= 0 \lor (-2x + 1).\]

Thus,

\[b'_m = \bigvee_x f(x) \]

\[= \bigvee_x \{0 \lor (-2x + 1)\} \]

\[= 1.\]

Therefore,

\[\text{not } A \Box R_m = \text{unknown}.\] (39)

We can obtain the consequences \( Ba' \) (cf. [8]), \( Bc', ..., B_l' \) in the same way as \( Bm' \), and thus we shall not discuss the details of how to obtain them. Table 1 summarizes the consequences inferred by all the inference methods (3)–(17) under the \( \max \odot \) composition.

We shall next discuss the fuzzy \textit{modus tollens} in (2). In the case of the fuzzy \textit{modus tollens}, it is assumed that \( \mu_B \) is a function onto \([0, 1]\). Because of the limitation of space, we shall investigate only the case of \( R_b \) of (12).
TABLE 1

Inference Results under Max-$\odot$ Composition (Case of Fuzzy Modus Ponens)

<table>
<thead>
<tr>
<th>$A$</th>
<th>very $A$</th>
<th>more or less $A$</th>
<th>not $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rm</td>
<td>B</td>
<td>$B$</td>
<td>$\frac{1}{4} \lor \mu_B$</td>
</tr>
<tr>
<td>Ra</td>
<td>$B$</td>
<td>$B$</td>
<td>${ \begin{cases} \mu_B + \frac{1}{4}, &amp; \mu_B \leq \frac{1}{4} \ \frac{1}{\sqrt{\mu_B}}, &amp; \mu_B &gt; \frac{1}{4} \end{cases}$</td>
</tr>
<tr>
<td>Rc</td>
<td>$B$</td>
<td>$B$</td>
<td>$B$</td>
</tr>
<tr>
<td>Rs</td>
<td>$B$</td>
<td>very $B$</td>
<td>more or less $B$</td>
</tr>
<tr>
<td>Rg</td>
<td>$B$</td>
<td>$B$</td>
<td>more or less $B$</td>
</tr>
<tr>
<td>Rgg</td>
<td>$B$</td>
<td>very $B$</td>
<td>more or less $B$</td>
</tr>
<tr>
<td>Rgs</td>
<td>$B$</td>
<td>$B$</td>
<td>more or less $B$</td>
</tr>
<tr>
<td>Rss</td>
<td>$B$</td>
<td>very $B$</td>
<td>more or less $B$</td>
</tr>
<tr>
<td>Rb</td>
<td>$B$</td>
<td>$B$</td>
<td>$\frac{1}{4} \lor \mu_B$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>$B$</td>
<td>$B$</td>
<td>more or less $B$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$B$</td>
<td>very $B$</td>
<td>more or less $B$</td>
</tr>
</tbody>
</table>

$R_e$ | $B$ | $B$ | $\{ \begin{cases} \frac{1}{4(1-\mu_B)}, & \mu_B \leq \frac{1}{2} \\ \mu_B, & \mu_B > \frac{1}{2} \end{cases}$ | unknown |
| $R_w$ | $B$ | $B$ | $\frac{1}{4} \lor \mu_B$ | $B \cup \text{not } B$ |
| $\square$ | unknown | unknown | unknown | unknown |

The consequence $Ab'$ is obtained [see (26)–(29)] by

\[
Ab' = R_b \square B',
\]

\[
\mu_{Ab'}(u) = \bigvee_v \left( \left[ \left[ 1 - \mu_A(u) \right] \lor \mu_B(v) \right] \circ \mu_B(v) \right).
\]

From the above assumption, this expression can be rewritten as

\[
a'_b = \bigvee_x \left( \left[ \left[ 1 - a \right] \lor x \right] \circ x' \right), \tag{40}
\]

\[
g(x) = \left[ \left[ 1 - a \right] \lor x \right] \circ x', \tag{41}
\]

where

\[
a'_b = \mu_{Ab'}(u), \quad a = \mu_A(u), \quad x = \mu_B(v), \quad x' = \mu_B(v). \tag{42}
\]
We shall show what the consequence $a'_b$ (or $A' b'$) will be when $B' = \text{not } B$, not very $B$, not more or less $B$, and $B$ under the max-$\odot$ composition.

(i) For $B' = \text{not } B$: When $B' = \text{not } B$, $x'$ becomes $1 - x$, from (42). Thus, $g(x)$ of (41) is given by

$$g(x) = [(1 - a) \lor x] \odot (1 - x)$$

$$= 0 \lor ((1 - a) \lor x + (1 - x) - 1)$$

$$= 0 \lor ((1 - a) - x \lor (x - x))$$

$$= 0 \lor (1 - a - x).$$

Therefore, from (40) we have $a'_b$ as

$$a'_b = \bigvee_x g(x)$$

$$= \bigvee_x (0 \lor (1 - a - x))$$

$$= 1 - a \quad \text{at} \quad x = 0.$$ 

It follows from this result that $a'_b = 1 - a$ at $x' = 1 - x$, that is, $A'b' = \text{not } A$ at $B' = \text{not } B$. Hence,

$$Rb \circ \text{not } B = \text{not } A. \quad (43)$$

This identity indicates the satisfaction of *modus tollens* by Rb under the max-$\odot$ composition. Note that Rb does not satisfy the *modus tollens* under the max-min composition [7].

(ii) For $B' = \text{not very } B$: Since $x' = 1 - x^2$, $g(x)$ will be

$$g(x) = [(1 - a) \lor x] \odot (1 - x^2)$$

$$= [0 \lor (1 - a - x^2)] \lor (x - x^2).$$

Therefore,

$$a'_b = \bigvee_x g(x)$$

$$= \bigvee_x ([0 \lor (1 - a - x^2)] \lor (x - x^2))$$

$$= \bigvee_x [0 \lor (1 - a - x^2)] \lor \bigvee_x (x - x^2)$$

$$= (1 - a) \lor \frac{1}{4}.$$
Hence

\[ \text{Rb not very } B = A', \]

where

\[ \mu_{A'} = \frac{1}{4} \lor (1 - \mu_A). \]  \hfill (44)

(iii) For \( B' = \text{not more or less } B \):

\[ g(x) = [(1 - a) \lor x] \odot (1 - \sqrt{x}) \]
\[ = 0 \lor (1 - a - \sqrt{x}), \]

\[ a'_b = \lor_x g(x) \]
\[ = \lor_x \{0 \lor (1 - a - \sqrt{x})\} \]
\[ = 1 - a \quad \text{at } x = 0. \]

Therefore,

\[ \text{Rb not more or less } B = \text{not } A. \]  \hfill (45)

(iv) For \( B' = B \):

\[ a'_b = \lor_x \{[(1 - a) \lor x] \odot x\} \]
\[ = \lor_x \{0 \lor (x - a) \lor (2x - 1)\} \]
\[ = 0 \lor \left[ \lor_x (x - a) \lor (2x - 1) \right] \]
\[ = 0 \lor (1 - a) \lor 1 \]
\[ = 1. \]

Thus,

\[ \text{Rb } B = \text{unknown}. \]
We can obtain the consequences $A_m', A_a$ (cf. [8]), $A_c', \ldots, A_{\Box}$ in the same way as $R_b'$. In Table 2 the inference results obtained by all the methods under the max-$\Theta$ composition are listed.

In the forms of fuzzy conditional inference (1) and (2), it seems according to our intuition that the relations between $A'$ in Ant 2 and $B'$ in Cons of the fuzzy *modus ponens* (1) ought to be satisfied as shown in Table 3 (cf. [4, 5]). Similarly, the relations between $B'$ in Ant 2 and $A'$ in Cons of the fuzzy *modus tollens* (2) ought to be satisfied as in Table 4.

In Table 5, the satisfaction ($\Theta$) or failure ($\times$) of each criterion of Tables 3 and 4 under each fuzzy inference method is indicated by use of the inference

<table>
<thead>
<tr>
<th>TABLE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inference Results under Max-$\Theta$ Composition (Case of Fuzzy Modus Tollens)</td>
</tr>
<tr>
<td>not $B$</td>
</tr>
<tr>
<td>Rm</td>
</tr>
<tr>
<td>Ra</td>
</tr>
<tr>
<td>Rc</td>
</tr>
<tr>
<td>Rs</td>
</tr>
<tr>
<td>Rg</td>
</tr>
<tr>
<td>RsG</td>
</tr>
<tr>
<td>RGG</td>
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<td>Rgs</td>
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<td>Rss</td>
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<tr>
<td>Rb</td>
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<td>$R_\alpha$</td>
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<tr>
<td>$R_\theta$</td>
</tr>
<tr>
<td>$R_{\Theta}$</td>
</tr>
<tr>
<td>$R_{\diamond}$</td>
</tr>
</tbody>
</table>
FUZZY CONDITIONAL INference

results in Tables 1 and 2. In order to compare the inference results under the max-\(\mho\) composition and the max-min composition, the inference results under the max-min composition are listed in Table 6 (cf. [7]).

From Tables 1, 2, and 5 it follows that all the inference methods except \(R_0\) can satisfy so-called modus ponens under the max-\(\mho\) composition, but only the methods \(R_c, R_s, \ldots, R_{ss}\) can satisfy the modus ponens under the max-min composition. Almost the same holds for modus tollens. Moreover, it is found that the majority of the methods can infer very reasonable consequences under the max-\(\mho\) composition, though we cannot always get reasonable consequences under the max-min composition, as shown in Table 6.

<table>
<thead>
<tr>
<th>Relation I (modus ponens)</th>
<th>(x) is (A') (Ant 2)</th>
<th>(y) is (B') (Cons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relation II-1</td>
<td>(x) is (very A)</td>
<td>(y) is (very B)</td>
</tr>
<tr>
<td>Relation II-2</td>
<td>(x) is (very A)</td>
<td>(y) is (B)</td>
</tr>
<tr>
<td>Relation III-1</td>
<td>(x) is (more or less A)</td>
<td>(y) is (more or less B)</td>
</tr>
<tr>
<td>Relation III-2</td>
<td>(x) is (more or less A)</td>
<td>(y) is (B)</td>
</tr>
<tr>
<td>Relation IV-1</td>
<td>(x) is (not A)</td>
<td>(y) is (unknown)</td>
</tr>
<tr>
<td>Relation IV-2</td>
<td>(x) is (not A)</td>
<td>(y) is (not B)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Relation V (modus tollens)</th>
<th>(y) is (B') (Ant 2)</th>
<th>(x) is (A') (Cons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relation VI-1</td>
<td>(y) is (not very B)</td>
<td>(x) is (not very A)</td>
</tr>
<tr>
<td>Relation VI-2</td>
<td>(y) is (not very B)</td>
<td>(x) is (not A)</td>
</tr>
<tr>
<td>Relation VII-1</td>
<td>(y) is (not more or less B)</td>
<td>(x) is (not more or less A)</td>
</tr>
<tr>
<td>Relation VII-2</td>
<td>(y) is (not more or less B)</td>
<td>(x) is (not A)</td>
</tr>
<tr>
<td>Relation VIII-1</td>
<td>(y) is (B)</td>
<td>(x) is (unknown)</td>
</tr>
<tr>
<td>Relation VIII-2</td>
<td>(y) is (B)</td>
<td>(x) is (A)</td>
</tr>
<tr>
<td>Relation I (modus ponens)</td>
<td>Relation II-1</td>
<td>Relation II-2</td>
</tr>
<tr>
<td>---------------------------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td>Ant 2</td>
<td>Cons</td>
<td>Rm</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>○</td>
</tr>
<tr>
<td>very A</td>
<td>very B</td>
<td>x</td>
</tr>
<tr>
<td>very A</td>
<td>B</td>
<td>○</td>
</tr>
<tr>
<td>more or less A</td>
<td>more or less B</td>
<td>x</td>
</tr>
<tr>
<td>more or less A</td>
<td>B</td>
<td>x</td>
</tr>
<tr>
<td>not A</td>
<td>unknown</td>
<td>o</td>
</tr>
<tr>
<td>not A</td>
<td>not B</td>
<td>x</td>
</tr>
<tr>
<td>not B</td>
<td>not A</td>
<td>○</td>
</tr>
<tr>
<td>not very B</td>
<td>not very A</td>
<td>x</td>
</tr>
<tr>
<td>not very B</td>
<td>not A</td>
<td>x</td>
</tr>
<tr>
<td>not more or less B</td>
<td>not more or less A</td>
<td>x</td>
</tr>
<tr>
<td>not more or less B</td>
<td>not A</td>
<td>o</td>
</tr>
<tr>
<td>B</td>
<td>unknown</td>
<td>x</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>x</td>
</tr>
<tr>
<td>Relation</td>
<td>Ant 2</td>
<td>Cons</td>
</tr>
<tr>
<td>----------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>I (modus ponens)</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>II-1</td>
<td>very A</td>
<td>very B</td>
</tr>
<tr>
<td>II-2</td>
<td>very A</td>
<td>B</td>
</tr>
<tr>
<td>III-1</td>
<td>more or less A</td>
<td>more or less B</td>
</tr>
<tr>
<td>III-2</td>
<td>more or less A</td>
<td>B</td>
</tr>
<tr>
<td>IV-1</td>
<td>not A</td>
<td>unknown</td>
</tr>
<tr>
<td>IV-2</td>
<td>not A</td>
<td>not B</td>
</tr>
<tr>
<td>V (modus tollens)</td>
<td>not B</td>
<td>not A</td>
</tr>
<tr>
<td>VI-1</td>
<td>not very B</td>
<td>not very A</td>
</tr>
<tr>
<td>VI-2</td>
<td>not very B</td>
<td>not A</td>
</tr>
<tr>
<td>VII-1</td>
<td>not more or less B</td>
<td>not more or less A</td>
</tr>
<tr>
<td>VII-2</td>
<td>not more or less B</td>
<td>not A</td>
</tr>
<tr>
<td>VIII-1</td>
<td>B</td>
<td>unknown</td>
</tr>
<tr>
<td>VIII-2</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

*aCf. [7].
4. SYLLOGISM BY EACH METHOD UNDER MAX-⊗ COMPOSITION

In this section we shall investigate a syllogism by each method under the max-⊗ composition □.

Let \( P_1, P_2 \) and \( P_3 \) be the fuzzy conditional propositions

\[
\begin{align*}
P_1 &: \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\
P_2 &: \text{If } y \text{ is } B \text{ then } z \text{ is } C. \\
P_3 &: \text{If } x \text{ is } A \text{ then } z \text{ is } C.
\end{align*}
\]

where \( A, B, \) and \( C \) are fuzzy sets in \( U, V, \) and \( W, \) respectively. If the proposition \( P_3 \) is deduced from the propositions \( P_1 \) and \( P_2 \)—that is, the following holds:

\[
\begin{align*}
P_1 &: \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\
P_2 &: \text{If } y \text{ is } B \text{ then } z \text{ is } C. \\
P_3 &: \text{If } x \text{ is } A \text{ then } z \text{ is } C.
\end{align*}
\]

—then it is said that a syllogism holds.

Let \( R(A, B), R(B, C), \) and \( R(A, C) \) be fuzzy relations in \( U \times V, V \times W, \) and \( U \times W, \) respectively, which are obtained from the propositions \( P_1, P_2, \) and \( P_3. \) If the following equality holds, the syllogism holds under the max-⊗ composition □:

\[
R(A, B) \square R(B, C) = R(A, C). \tag{46}
\]

That is to say,

\[
\begin{align*}
P_1 &: \text{If } x \text{ is } A \text{ then } y \text{ is } B \rightarrow R(A, B) \\
P_2 &: \text{If } y \text{ is } B \text{ then } z \text{ is } C \rightarrow R(B, C) \\
P_3 &: \text{If } x \text{ is } A \text{ then } z \text{ is } C \leftarrow R(A, B) \square R(B, C)
\end{align*}
\]

The membership function of \( R(A, B) \square R(B, C) \) is given by

\[
\mu_{R(A, B) \square R(B, C)}(u, w) = \bigvee_v \{ \mu_{R(A, B)}(u, v) \odot \mu_{R(B, C)}(v, w) \}. \tag{48}
\]

Now we shall obtain \( R(A, B) \square R(B, C) \) under each fuzzy inference method and show whether the syllogism holds or not. In the discussion of the syllogism it is assumed that the membership function \( \mu_B \) of the fuzzy set \( B \) is a function onto \([0, 1]\).
We shall discuss only the case of $R_\Delta$ of (13). The membership functions of the fuzzy relations $R_\Delta(A,B)$ and $R_\Delta(B,C)$ are obtained from the propositions $P_1$ and $P_2$ by using (13).

$$\begin{align*}
\mu_{R_\Delta(A,B)}(u,v) &= \mu_A(u) \rightarrow \mu_B(v) \\
&= \begin{cases}
1, & \mu_A(u) \leq \mu_B(v), \\
\frac{\mu_B(v)}{\mu_A(u)}, & \mu_A(u) > \mu_B(v),
\end{cases} \\
&= \begin{cases}
1 \land \frac{\mu_B(v)}{\mu_A(u)}, & \mu_A(u) > 0, \\
1, & \mu_A(u) = 0.
\end{cases}
\end{align*}$$

(49)

$$\begin{align*}
\mu_{R_\Delta(B,C)}(v,w) &= \mu_B(v) \rightarrow \mu_C(w) \\
&= \begin{cases}
1, & \mu_B(v) \leq \mu_C(w), \\
\frac{\mu_C(w)}{\mu_B(v)}, & \mu_B(v) > \mu_C(u),
\end{cases} \\
&= \begin{cases}
1 \land \frac{\mu_C(w)}{\mu_B(v)}, & \mu_B(v) > 0, \\
1, & \mu_B(v) = 0.
\end{cases}
\end{align*}$$

(50)

Then the membership functions of the max-$\odot$ composition of $R_\Delta(A,B)$ and $R_\Delta(B,C)$ will be given by

$$\begin{align*}
\mu_{R_\Delta(A,B) \odot R_\Delta(B,C)}(u,w) &= \bigvee_v \left\{ \left[ \mu_A(u) \rightarrow \mu_B(v) \right] \odot \left[ \mu_B(v) \rightarrow \mu_C(w) \right] \right\}.
\end{align*}$$

(51)

Under the assumption that $\mu_B$ is a function onto $[0,1]$, (51) is rewritten as

$$\begin{align*}
d &= \bigvee_x \left\{ [a \rightarrow x] \odot [x \rightarrow c] \right\},
\end{align*}$$

(52)

where

$$\begin{align*}
d &= \mu_{R_\Delta(A,B) \odot R_\Delta(B,C)}(u,w), \quad a = \mu_A(u), \quad x = \mu_B(v), \quad c = \mu_C(w)
\end{align*}$$

(53)
and

\[
\begin{align*}
\Delta a \rightarrow x &= \begin{cases} 
1 \wedge \frac{x}{a}, & a > 0, \\
1, & a = 0,
\end{cases} \\
\Delta x \rightarrow c &= \begin{cases} 
1 \wedge \frac{c}{x}, & x > 0, \\
1, & x = 0.
\end{cases}
\end{align*}
\]

Then, \(\Delta [a \rightarrow x] \ominus [x \rightarrow c]\) is given as

\[
\begin{align*}
\Delta [a \rightarrow x] \ominus [x \rightarrow c] = & \begin{cases} 
1 \wedge \frac{x}{a} \wedge \frac{c}{x} \wedge \left(\frac{x}{a} + \frac{c}{x} - 1\right), & a, x > 0, \\
1, & a, x = 0,
\end{cases} \\
& (a = 0, x > 0) \text{ or } (a > 0, x = 0).
\end{align*}
\] (54)

When \(a > c\), the expression (54) is represented by the solid line in Figure 2(a) with parameters \(a\) and \(c\). The maximum value of this line is \(c/a\) at \(x = a\) and \(c\). Thus, we have \(d\) of (52) as

\[
d = \frac{c}{a}, \quad a > c.
\] (55)

---

![Graphs](graph.png)

(a) At \(a > c\)  
(b) At \(a \leq c\)

Fig. 2. \(\Delta [a \rightarrow x] \ominus [x \rightarrow c]\) of (54) (solid line).
FUZZY CONDITIONAL INFERENCE

On the other hand, when \( a \leq c \), (54) is shown by the solid line in Figure 2(b), whose maximum value is 1. Thus,

\[
d = 1, \quad a \leq c. \tag{56}
\]

From (55) and (56), \( d \) is given by

\[
d = \begin{cases} 
1, & a \leq c, \\
\frac{c}{a}, & a > c,
\end{cases}
\]

which leads to

\[
\mu_{R_\Delta(A, B) \Box R_\Delta(B, C)}(u, v) = \begin{cases} 
1, & \mu_A(u) \leq \mu_C(w), \\
\mu_C(w), & \mu_A(u) > \mu_C(w)
\end{cases}
\]

\[
= \mu_A(u) \rightarrow \mu_C(w)
\]

\[
= \mu_{R_\Delta(A, C)}(u, w). \tag{57}
\]

Thus, we have

\[
R_\Delta(A, B) \Box R_\Delta(B, C) = R_\Delta(A, C). \tag{58}
\]

Therefore, the syllogism holds for \( R_\Delta \) under the \( \max-\Box \) composition \( \Box \). Note that \( R_\Delta \) does not satisfy the syllogism under the \( \max-min \) composition \([7]\).

In the same way, we can obtain \( R(A, B) \Box R(B, C) \) by the other methods; the results are as follows:

\[
Rm(A, B) \Box Rm(B, C)
\]

\[
= \int_{U \times W} \left[ \mu_A(u) + \mu_C(w) - 1 \right] \vee \left[ 1 - \mu_A(u) \right] / (u, w)
\]

\[
= Rm(A, C) - \left[ \int_{U \times W} \left[ \mu_A(u) \wedge \mu_C(w) \right] \vee \left[ 1 - \mu_A(u) \right] / (u, w) \right], \tag{59}
\]
\[
\begin{align*}
\text{Ra}(A, B) \square \text{Ra}(B, C) &= \int_{U \times W} 1 \land \left[1 - \mu_A(u) + \mu_C(w)\right] / (u, w) \\
&= \text{Ra}(A, C), \\
\text{Rc}(A, B) \square \text{Rc}(B, C) &= \int_{U \times W} 0 \lor \left[\mu_A(u) + \mu_C(w) - 1\right] / (u, w) \\
&= \text{Rc}(A, C) \left(= \int_{U \times W} \mu_A(u) \land \mu_C(w) / (u, w)\right), \\
\text{Rs}(A, B) \square \text{Rs}(B, C) &= \int_{U \times W} \mu_A(u) \Rightarrow \mu_C(w) / (u, w) \\
&= \text{Rs}(A, C), \\
\text{Rg}(A, B) \square \text{Rg}(B, C) &= \int_{U \times W} \mu_A(u) \Rightarrow \mu_C(w) / (u, w) \\
&= \text{Rg}(A, C), \\
\text{Rsg}(A, B) \square \text{Rsg}(B, C) &= \int_{U \times W} \left[\mu_A(u) \Rightarrow \mu_C(w)\right] \\
&\land \left[1 - \mu_A(u) \Rightarrow 1 - \mu_C(w)\right] / (u, w) \\
&= \text{Rsg}(A, C), \\
\text{Rgg}(A, B) \square \text{Rgg}(B, C) &= \int_{U \times W} \left[\mu_A(u) \Rightarrow \mu_C(w)\right] \\
&\land \left[1 - \mu_A(u) \Rightarrow 1 - \mu_C(w)\right] / (u, w) \\
&= \text{Rgg}(A, C), \\
\text{Rgs}(A, B) \square \text{Rgs}(B, C) &= \int_{U \times W} \left[\mu_A(u) \Rightarrow \mu_C(w)\right] \\
&\land \left[1 - \mu_A(u) \Rightarrow 1 - \mu_C(w)\right] / (u, w) \\
&= \text{Rgs}(A, C), \\
\text{Rss}(A, B) \square \text{Rss}(B, C) &= \int_{U \times W} \left[\mu_A(u) \Rightarrow \mu_C(w)\right] \\
&\land \left[1 - \mu_A(u) \Rightarrow 1 - \mu_C(w)\right] / (u, w) \\
&= \text{Rss}(A, C),
\end{align*}
\]
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\[ R_b(A, B) \square R_b(B, C) = \int_{U \times W} [1 - \mu_A(u)] \vee \mu_C(w)/(u, w) \]
\[ = R_b(A, C), \] \hspace{1cm} (68)

\[ R_\triangle(A, B) \square R_\triangle(B, C) = \int_{U \times W} \mu_A(u) \rightarrow \mu_C(w)/(u, w) \]
\[ = R_\triangle(A, C), \] \hspace{1cm} (69)

\[ R_\tri(A, B) \square R_\tri(B, C) = \int_{U \times W} \mu_A(u) \rightarrow \mu_C(w)/(u, w) \]
\[ = R_\tri(A, C), \] \hspace{1cm} (70)

\[ R_\ast(A, B) \square R_\ast(B, C) = \int_{U \times W} [1 - \mu_A(u)] \vee \mu_C(w)/(u, w) \]
\[ = R_\ast(A, C) \left( \frac{1}{\int_{U \times W} 1 - \mu_A(u)} + \mu_A(u) \mu_C(w)/(u, w) \right), \] \hspace{1cm} (71)

\[ R_\ast(A, B) \square R_\ast(B, C) = \int_{U \times W} [\mu_A(u) + \mu_C(w) - 1] \vee [1 - \mu_A(u) - \mu_C(w)] \]
\[ \vee [\mu_C(w) - \mu_A(u)]/(u, w) = R_\ast(A, C) \]
\[ \left( = \int_{U \times W} [\mu_A(u) \wedge \mu_C(w)] \right) \]
\[ \vee [1 - \mu_A(u) \wedge 1 - \mu_C(w)] \]
\[ \vee [1 - \mu_A(u) \wedge \mu_C(w)]/(u, w), \] \hspace{1cm} (72)

\[ R_\Box(A, B) \square R_\Box(B, C) = \int_{U \times W} \mu_A(u) \rightarrow \mu_C(w)/(u, w) \]
\[ = R_\Box(A, C). \] \hspace{1cm} (73)
TABLE 7
Satisfaction of Syllogism under Max-\(\bigodot\) Composition and Max-Min Composition

<table>
<thead>
<tr>
<th></th>
<th>Rm</th>
<th>Ra</th>
<th>Rc</th>
<th>Rs</th>
<th>Rg</th>
<th>Rsg</th>
<th>Rgs</th>
<th>RsS</th>
<th>Rb</th>
<th>R(\bigodot)</th>
<th>R(\bigtriangleup)</th>
<th>R(\ast)</th>
<th>R(\bigcirc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max-(\bigodot) composition</td>
<td>(\times)</td>
<td>(\bigcirc)</td>
<td>(\times)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\ast)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
</tr>
<tr>
<td>Max-min composition</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\bigcirc)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\bigcirc)</td>
</tr>
</tbody>
</table>

Using these results, the satisfaction (\(\bigcirc\)) or failure (\(\times\)) of syllogism by each method under the max-\(\bigodot\) composition is listed in Table 7. This table also contains the results under the max-min composition (cf. [7]).

It follows from Table 7 that the methods Ra, Rb, R\(\bigtriangleup\), and R\(\ast\) can satisfy the syllogism under the max-\(\bigodot\) composition, though they do not satisfy it under the max-min composition. But the converse holds for Rc.

5. CONCLUSION

We have shown that, when the max-\(\bigodot\) composition is used in the compositional rule of inference, the majority of fuzzy inference methods can lead to very reasonable consequences which coincide with our intuition with respect to several criteria such as *modus ponens*, *modus tollens*, and syllogism.

It will be of interest to apply the max-\(\bigodot\) composition to fuzzy inferences which are of the more complicated form, such as

If \(x\) is \(A\) then \(y\) is \(B\) else \(y\) is \(C\).
\(x\) is \(A'\).
\(y\) is \(D\).
If \(x\) is \(A_1\) then \(y\) is \(B_1\) else
if \(x\) is \(A_2\) then \(y\) is \(B_2\) else
: :
if \(x\) is \(A_n\) then \(y\) is \(B_n\).
\(x\) is \(A'\).
\(y\) is \(B'\).

These results will be presented in subsequent papers.

*This work was attained during the author’s stay (November 1980 – August 1981) at RWTH Aachen, West Germany, with the assistance of the Alexander von*
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Humboldt Foundation. He acknowledges the invaluable help of Professor H.-J. Zimmermann and the members of fuzzy research group at RWTH Aachen.

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Received 2 July 1982